

Separation index of graphs and stacked 2-spheres

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Abstract

In recent work on combinatorial criteria for tight triangulations, Bagchi and Datta introduced the sigma-vector $(\sigma_0(X), \dots, \sigma_d(X))$ of a d -dimensional simplicial complex X . We call $\sigma_0(G)$ of a graph G as the *separation index* of G and denote it by $s(G)$. We show that if G is the 1-skeleton of an n -vertex triangulation X of the 2-sphere S^2 , then $s(G) \leq (n-8)(n+1)/20$, with equality if and only if X is a stacked 2-sphere. Using this characterization of stacked 2-spheres, we settle the outstanding 3-dimensional case of the Lutz-Sulanke-Swartz conjecture that ‘tight neighborly triangulated manifolds are tight’. For dimension $d \geq 4$, the conjecture follows from results of Novik-Swartz and Effenberger.

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1 Introduction and results

For a graph G , we investigate its measure of “average” disconnectivity $s(G)$, which we call its *separation index*. This measure already appears as part of the sigma- and mu-vectors for simplicial complexes introduced by Bagchi and Datta [3] for studying tight triangulations (see Definition 2.10 below). Roughly, $s(G)$ is the weighted average of the number of components over all induced subgraphs of G .

While we believe it is interesting to study $s(G)$ for graphs in general, in this paper we consider the case when G is the 1-skeleton of a triangulation of the 2-sphere S^2 . For a simplicial complex X , $s(X)$ will denote the separation index of the 1-skeleton of X . We show:

Theorem 1.1. *Let X be an n -vertex triangulation of the 2-sphere S^2 . Then $s(X) \leq (n-8)(n+1)/20$, where equality occurs if and only if X is a stacked sphere.*

A striking implication of Theorem 1.1 is that all tight neighborly 3-manifolds have stacked spheres as vertex-links, i.e., they belong to Walkup's class $\mathcal{K}(3)$ of triangulated 3-manifolds. Indeed, we prove:

Theorem 1.2. *Let X be a triangulated 3-manifold. If X is tight neighborly then X is a neighborly member of $\mathcal{K}(3)$.*

Novik and Swartz [8] proved similar result for dimension $d \geq 4$. More precisely, they proved that a tight neighbourly triangulated d -manifold belongs to $\mathcal{K}(d)$ for $d \geq 4$. In [7], Lutz, Sulanke and Swartz conjectured that, for $d \geq 3$, all tight neighborly triangulated d -manifolds are tight. Using Novik-Swartz's result, Effenberger [5] proved this conjecture for $d \geq 4$. As a consequence of Theorem 1.2 and Proposition 2.5 below of Bagchi-Datta, we prove the conjecture for the remaining case, that is:

Corollary 1.3. *Let X be a tight neighborly triangulated 3-manifold. Then X is tight.*

Remark 1.4. Although the converse of Theorem 1.2 is true in dimensions $d \geq 4$ (see Corollary 2.3), it is not true in dimension three (see Remark 3.5).

Remark 1.5. For $d \geq 3$, equality in the lower bound theorem [6] characterizes stacked d -spheres among triangulated d -spheres. That is, an n vertex triangulated d -sphere S is a stacked d -sphere if and only if $f_1(S) = (d+1)n - \binom{d+2}{2}$. Theorem 1.1 in this paper gives a characterization of stacked 2-spheres among triangulated 2-spheres.

2 Preliminaries and basic results

2.1 Simplicial complexes and graphs

All simplicial complexes considered here are finite and abstract. The vertex set of a simplicial complex X is denoted by $V(X)$. By a *triangulation* of a space M , we mean a simplicial complex X whose geometric carrier is M . By a *triangulated d -manifold* we mean a triangulation of a topological manifold of dimension d . The boundary complex of a $(d+1)$ -simplex is a triangulated d -manifold with $d+2$ vertices and triangulates the d -sphere S^d . It is called the *standard d -sphere* and is denoted by S_{d+2}^d . A simplicial complex of dimension d is called *pure* if all its maximal faces are d -dimensional. For a simplicial complex X , and $A \subseteq V(X)$, $X[A]$ denotes the simplicial complex consisting of all faces of X which are contained in A . We say that $X[A]$ is the subcomplex of X *induced* by the set A .

For a finite set A , let $\text{Cl}(A)$ denote the simplicial complex consisting of all subsets of A . The *link* of a vertex x in a simplicial complex X is defined to be the subcomplex $\text{lk}_X(x) := \{\alpha \in X : x \notin \alpha, \alpha \cup \{x\} \in X\}$. For $k \leq \dim(X)$, we define $\text{skel}_k(X) := \{\alpha \in X : |\alpha| \leq k+1\}$ to be the *k -skeleton* of the simplicial complex X .

For a d -dimensional simplicial complex X , the vector (f_0, f_1, \dots, f_d) is called its *f -vector*, where $f_i = f_i(X)$ is the number of i -dimensional faces of X . We will call a simplicial complex *neighborly* if $f_1 = \binom{f_0}{2}$, i.e., any two vertices form an edge.

Unless the field is explicitly mentioned, the homologies and Betti numbers are considered w.r.t. \mathbb{Z}_2 , but the arguments hold for an arbitrary field \mathbb{F} , when the manifold

is \mathbb{F} -orientable. So, $H_i(X) = H_i(X; \mathbb{Z}_2)$ and $\beta_i(X) = \beta_i(X; \mathbb{Z}_2)$ for all $i \geq 0$ and for all simplicial complexes X .

All graphs considered here are finite and simple. A standard reference for basic terminology on graphs is [4]. For a graph G , $V(G)$ and $E(G)$ will denote its vertex-set and edge-set respectively. For $v \in V(G)$, $d_G(v)$ denotes the *degree* of v in G . The set of neighbors of v in G is denoted by $N_G(v)$, or just $N(v)$ when the ambient graph is clear from the context. For $n \geq 3$, an n -cycle with edges $u_1u_2, \dots, u_{n-1}u_n, u_nu_1$ is denoted by $C_n(u_1, u_2, \dots, u_n)$. A graph is called *planar* if it can be embedded in a plane (or 2-sphere) without the edges intersecting in an interior point. The following are well known.

Lemma 2.1. *Let G be a planar graph. Then,*

- (a) *G does not contain K_5 as a subgraph;*
- (b) $|E(G)| \leq 3|V(G)| - 6$.

2.2 Stacked spheres

Let X be a pure d -dimensional simplicial complex and let $x \notin V(X)$. We say that Y is obtained from X by *starring* the vertex x in the d -face σ of X , if $Y = (X \setminus \{\sigma\}) \cup \{\tau \cup \{x\} : \tau \subset \sigma\}$. A simplicial complex is called a *stacked d -sphere* if it is obtained from S_{d+2}^d by a finite sequence of starring operations. It is clear that a stacked d -sphere triangulates the d -sphere S^d . We know that a stacked d -sphere has at least two vertices of degree $d+1$, i.e., whose links are standard $(d-1)$ -spheres with $d+1$ vertices (cf. Lemma 4.3(b) in [1]).

In [10], Walkup defined the class $\mathcal{K}(d)$ of triangulated d -manifolds whose vertex-links are stacked $(d-1)$ -spheres.

2.3 Tight neighborly manifolds

The following result by Novik and Swartz [8] gives an upper bound on the first Betti number of a triangulated d -manifold depending on its 1-skeleton. They prove:

Proposition 2.2 (Novik, Swartz). *Let X be a connected triangulated d -manifold.*

- (a) *If $d \geq 3$ then $\binom{d+2}{2}\beta_1(X) \leq f_1(X) - (d+1)f_0(X) + \binom{d+2}{2}$.*
- (b) *Further, if $d \geq 4$ and $\binom{d+2}{2}\beta_1(X) = f_1(X) - (d+1)f_0(X) + \binom{d+2}{2}$ then $X \in \mathcal{K}(d)$.*

In [7], Lutz, Sulanke and Swartz observed that Proposition 2.2 implies:

Corollary 2.3. *Let X be a connected triangulated d -manifold. If $d \geq 3$, then*

$$\binom{d+2}{2}\beta_1(X) \leq \binom{f_0(X) - d - 1}{2}. \quad (1)$$

Moreover for $d \geq 4$, equality holds if and only if X is a neighborly member of $\mathcal{K}(d)$.

For $d \geq 3$, a triangulated d -manifold is called *tight neighborly* if it satisfies (1) with equality. From part (a) of Proposition 2.2 and the trivial inequality $f_1(X) \leq \binom{f_0(X)}{2}$ it can be seen that tight neighborly triangulated manifolds are neighborly.

2.4 Tight triangulations

A d -dimensional connected simplicial complex X is said to be *tight* if for all $A \subseteq V(X)$, the homology maps induced by the inclusion map, namely $H_i(X[A]; \mathbb{Z}_2) \rightarrow H_i(X; \mathbb{Z}_2)$, are injective for all $0 \leq i \leq d$. Examples of tight triangulations are extremely rare, and have so far evaded complete combinatorial characterization. We know the following:

Proposition 2.4 (Effenberger [5]). *For $d \neq 3$, the neighborly members of $\mathcal{K}(d)$ are tight.*

Proposition 2.5 (Bagchi, Datta [3]). *If X is a neighborly member of $\mathcal{K}(3)$, then X is tight if and only if $\beta_1(X) = (f_0(X) - 4)(f_0(X) - 5)/20$.*

In this paper we settle the following conjecture completely.

Conjecture 2.6 (Lutz, Sulanke, Swartz [7]). *Tight-neighborly triangulations are tight.*

2.5 Bistellar flips

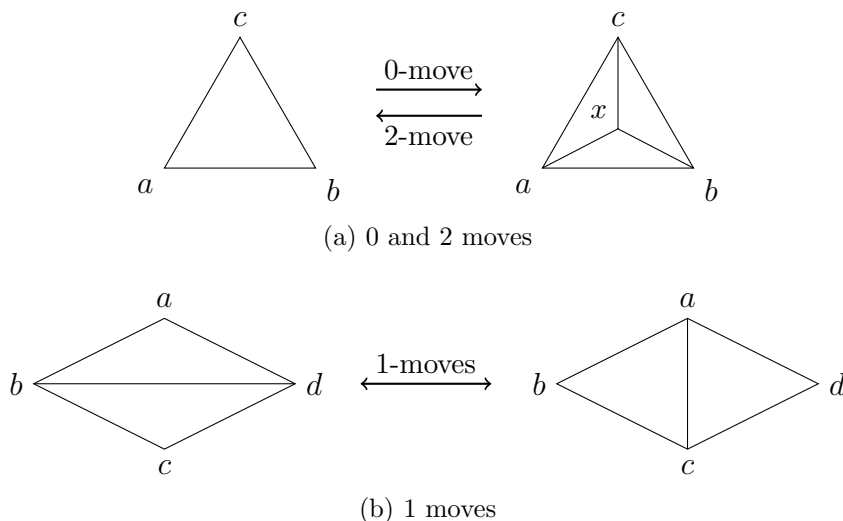


Figure 1: Bistellar moves

Bistellar flips or *Pachner moves* are ways of replacing a combinatorial triangulation of a piecewise linear manifold with another such triangulation of the same manifold. In dimension two, we have the following bistellar moves:

- (a) (**Bistellar 0-, 2-moves:**) Let X be a two-dimensional pure simplicial complex. If Y is obtained from X by starring a vertex x in the face abc of X , we say that Y is obtained from X by the bistellar 0-move $abc \mapsto x$. We also say that X is obtained from Y by the bistellar 2-move $x \mapsto abc$ (see Figure 1(a)).

- (b) (**Bistellar 1-moves:**) Let X be a pure two-dimensional simplicial complex and let abd and bdc be two adjacent faces of X such that ac is not an edge. If $Y = (X \setminus \{abd, bdc, bd\}) \cup \{abc, acd, ac\}$ then Y and X triangulate the same space. We say that Y is obtained from X by the bistellar 1-move $bd \mapsto ac$. Observe that, in this case, X is obtained from Y by the bistellar move $ac \mapsto bd$ (see Figure 1(b)).

As a consequence of Pachner's classical theorem we have,

Proposition 2.7 (Pachner [9]). *Any triangulation of S^2 can be obtained from the standard 2-sphere S_4^2 by a finite sequence of bistellar 0-, 1- and 2-moves.*

Definition 2.8 (1A-move and 1B-move). Let X be a pure two-dimensional simplicial complex. Let a, b, c, d be vertices of X such that abd, bdc are two adjacent faces and ac is not an edge. Then the 1-move $bd \mapsto ac$ is said to be of type 1A if one of a and c is a vertex of degree 3 in X . Similarly the 1-move $bd \mapsto ac$ is said to be of type 1B if the degree of one of a and c in X is precisely 4 and the degree of the other one is at least 4.

We will need the following slightly stronger version of Proposition 2.7 for our purpose. In fact, the arguments in the proof of Lemma 2.9 will be used again later in the proof of Theorem 1.2.

Lemma 2.9. *Let X be a triangulation of S^2 . Then X can be obtained from S_4^2 by a finite sequence of bistellar 0-moves, 1A-moves and 1B-moves.*

Proof. We proceed by induction on the number of vertices in X , i.e., on $n = f_0(X)$. Clearly, the lemma is true for $n = 4$. So, assume that $n \geq 5$ and the lemma is true for all triangulations Y of S^2 with $f_0(Y) < n$. Let $f_0(X) = n$. Since the 1-skeleton of X is a planar graph, X must have a vertex of degree at most 5. We have the following cases:

Case 1: X has a vertex of degree 3. Let x be the vertex of degree 3 in X . Let a, b, c be the neighbors of x in X . Clearly, the triangles xab, xbc, xac are faces in X . However abc cannot be a face of X , otherwise $X[\{a, b, c, d\}] \cong S_4^2$. Now consider $Y := X[V(X) \setminus \{x\}] \cup \{abc\}$. It is easily seen that X is obtained from Y by a 0-move. The lemma now follows by the induction hypothesis.

Case 2: X has a vertex of degree 4. Let x be a vertex of degree 4 and let a, b, c, d be its neighbors such that $\text{lk}_X(x)$ is the 4-cycle $C_4(a, b, c, d)$. Since K_5 is not planar, we have $K_5 \not\subseteq X$. Hence there is a pair of non-adjacent vertices among a, b, c, d . Assume ac is a non-edge. Define $Y := X[V(X) \setminus \{x\}] \cup \{abc, acd, ac\}$. Then X can be obtained from Y by a 0-move followed by a 1A-move as illustrated in figure 2(a). The result then follows by invoking the induction hypothesis for Y .

Case 3: All vertices of X have degree 5 or more. Let x be a vertex of degree 5. Let a, b, c, d, e be neighbors of x such that $\text{lk}_X(x)$ is the cycle $C_5(a, b, c, d, e)$. Since the induced subgraph on vertices $\{x, a, b, c, d, e\}$ is planar, by Euler's bound (Lemma 2.1(b)) it can have at most $3 \times 6 - 6 = 12$ edges. It follows then that at least one vertex among a, b, c, d, e has two non-neighbors. Let a be such a vertex, with nonedges ac and ad . Consider $Y := X[V(X) \setminus \{x\}] \cup \{abc, acd, ade, ac, ad\}$. Then X can be obtained

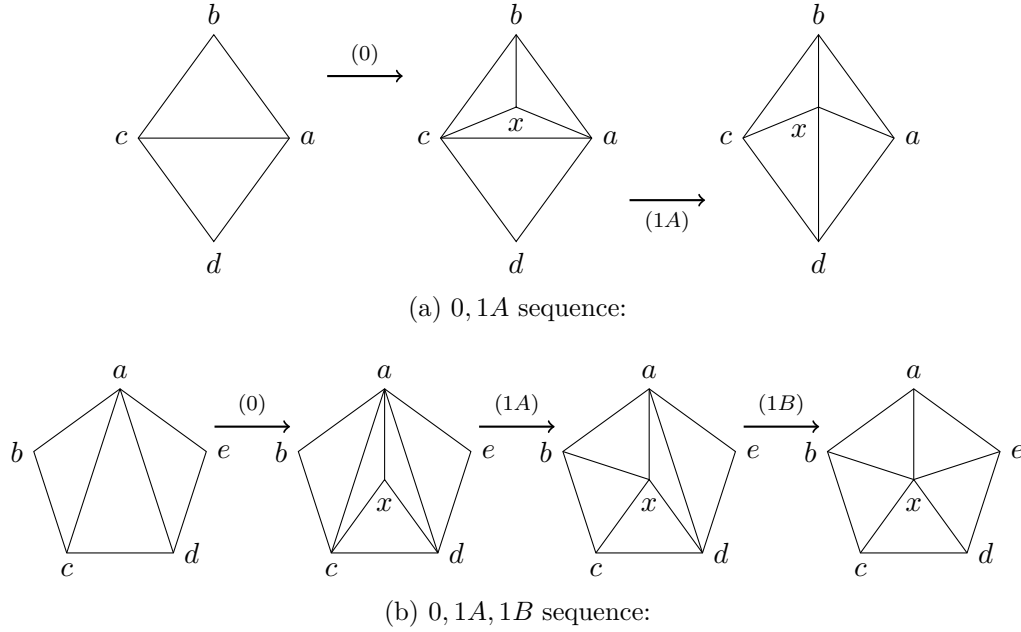


Figure 2: Illustration for Lemma 2.9

from Y by a sequence of 0-move, 1A-move and 1B-move as illustrated in figure 2(b). The lemma follows by using the induction hypothesis for Y . \square

2.6 The sigma-vector and mu-vector

For any set V and any integer $i \geq 0$, the collection of all i -element subsets of V will be denoted by $\binom{V}{i}$. Let X be a simplicial complex of dimension d . As usual, $\tilde{\beta}_i(X)$ denotes the reduced i^{th} homology of X . Thus, $\tilde{\beta}_0(X) = \beta_0(X; \mathbb{Z}_2) - 1$ and $\tilde{\beta}_i(X) = \beta_i(X; \mathbb{Z}_2)$ for $i > 0$. We recall the following definitions from [3].

Definition 2.10. Let X be a d -dimensional simplicial complex on m vertices. The *sigma-vector* $(\sigma_0, \sigma_1, \dots, \sigma_d)$ of X is defined by

$$\sigma_i = \sigma_i(X) = \sum_{A \subseteq V(X)} \frac{\tilde{\beta}_i(X[A])}{\binom{m}{|A|}}, \quad 0 \leq i \leq d. \quad (2)$$

The *mu-vector* $(\mu_0, \mu_1, \dots, \mu_d)$ of X is defined by

$$\begin{aligned} \mu_0 &= \mu_0(X) = 1, \\ \mu_i &= \mu_i(X) = \delta_{i1} + \frac{1}{m} \sum_{x \in V(X)} \sigma_{i-1}(\text{lk}_X(x)), \quad 1 \leq i \leq d. \end{aligned} \quad (3)$$

The following result follows from Theorem 2.6 in [3].

Proposition 2.11. *Let X be a neighborly simplicial complex of dimension d . Then $\beta_i \leq \mu_i$ for all $0 \leq i \leq d$.*

3 Separation index of a graph

For a graph G , let $q(G)$ denote the number of connected components of G . We know that $\beta_0(X)$ is the number of connected components of X , which is same as the number of connected components in the 1-skeleton of X . Thus, we see that $\sigma_0(X)$ is (roughly) a weighted average of the number of connected components over all induced subgraphs of the 1-skeleton of X . This motivates us to define:

Definition 3.1. Let G be a graph on n vertices. We define the *separation index* of G to be $s(G)$, given by

$$s(G) := \sum_{A \subseteq V(G)} \frac{q(G[A]) - 1}{\binom{n}{|A|}} = \sum_{i=0}^n s_i(G), \quad (4)$$

where

$$s_i(G) := \frac{1}{\binom{n}{i}} \sum_{\substack{A \subseteq V(G) \\ |A|=i}} (q(G[A]) - 1) \text{ for } 0 \leq i \leq n.$$

So, $s_i(G)$ is one less than the average number of components in an induced subgraph with i vertices.

It is easily seen that for any graph G , we have $-1 \leq s(G) \leq (n+1)(n-2)/2$, with the lower and upper bounds attained by the complete graph K_n and its complement respectively.

Lemma 3.2. Let X be obtained from Y by a bistellar 0-move. If Y triangulates S^2 and $f_0(Y) = n$ then $s(X) = (n+2)s(Y)/(n+1) + (n+2)/20$.

Proof. Let X be obtained from Y by starring the vertex v_0 in the face $v_1v_2v_3$. Let $G = \text{skel}_1(Y)$ and $H = \text{skel}_1(X)$. Then H is obtained from G by introducing a new vertex v_0 and joining it to vertices v_1, v_2, v_3 . We recall that v_1, v_2, v_3 are mutually adjacent in G . Let $V := V(Y)$. Now from (2) we have

$$s(H) = \sum_{S \subseteq V} \frac{q(H[S]) - 1}{\binom{n+1}{|S|}} + \sum_{S \subseteq V} \frac{q(H[S \cup \{v_0\}]) - 1}{\binom{n+1}{|S|+1}}. \quad (5)$$

We note that $q(H[S]) = q(G[S])$ for $S \subseteq V$, $q(H[S \cup \{v_0\}]) = q(G[S])$ when S has a non-empty intersection with $N(v_0) = \{v_1, v_2, v_3\}$ (which we denote by $S \leftrightarrow N(v_0)$), and $q(H[S \cup \{v_0\}]) = q(G[S]) + 1$ when S does not intersect $N(v_0)$ (which we denote by $S \nleftrightarrow N(v_0)$). We split the second summation in (5) to get

$$\begin{aligned}
s(H) &= \sum_{S \subseteq V} \frac{q(G[S]) - 1}{\binom{n+1}{|S|}} + \sum_{S \leftrightarrow N(v_0)} \frac{q(G[S])}{\binom{n+1}{|S|+1}} + \sum_{S \leftrightarrow N(v_0)} \frac{q(G[S]) - 1}{\binom{n+1}{|S|+1}} \\
&= \sum_{S \subseteq V} \frac{q(G[S]) - 1}{\binom{n+1}{|S|}} + \sum_{S \subseteq V} \frac{q(G[S]) - 1}{\binom{n+1}{|S|+1}} + \sum_{S \leftrightarrow N(v_0)} \frac{1}{\binom{n+1}{|S|+1}} \\
&= \sum_{S \subseteq V} \frac{q(G[S]) - 1}{\binom{n+1}{|S|}} \left[\frac{n - |S| + 1}{n + 1} + \frac{|S| + 1}{n + 1} \right] + \sum_{S \leftrightarrow N(v_0)} \frac{1}{\binom{n+1}{|S|+1}} \\
&= \frac{n + 2}{n + 1} \times s(G) + \sum_{S \leftrightarrow N(v_0)} \frac{1}{\binom{n+1}{|S|+1}}. \tag{6}
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{S \leftrightarrow N(v_0)} \frac{1}{\binom{n+1}{|S|+1}} &= \sum_{j=0}^{n-3} \frac{\binom{n-3}{j}}{\binom{n+1}{j+1}} = \frac{6 \times (n-3)!}{(n+1)!} \sum_{j=0}^{n-3} \frac{(n-j)!}{3!(n-j-3)!} (j+1) \\
&= \frac{6 \times (n-3)!}{(n+1)!} \times \binom{n+2}{5} = \frac{n+2}{20}.
\end{aligned}$$

In the above, we have used the fact that $\sum_{j=0}^{n-3} \binom{n-j}{3} \times (j+1) = \binom{n+2}{5}$. This is because the first summation is the number of ways of choosing 5 objects from $n+2$ ordered objects, where the fourth object is the $(n-j+1)^{th}$ object, the first three come from the first $n-j$ objects, and the last one comes from the remaining $j+1$ objects. The lemma now follows from (6). \square

Corollary 3.3. *Let X be a stacked 2-sphere. If $f_0(X) = n$ then $s(X) = \frac{(n-8)(n+1)}{20}$.*

Proof. Since an n -vertex stacked 2-sphere is obtained from the standard 2-sphere S_4^2 by a sequence of $n-4$ bistellar 0-moves, the corollary follows by induction and Lemma 3.2. \square

We now prove Theorem 1.1.

Proof of Theorem 1.1. We have already shown that when X is a stacked 2-sphere on n vertices, $s(X) = (n-8)(n+1)/20$. It remains to show that $s(X) < (n-8)(n+1)/20$ when X is not a stacked 2-sphere. We proceed by induction. For $n = 4, 5$ the result is vacuously true as all triangulations of S^2 on at most 5 vertices are stacked. Assume that $n \geq 5$ and that the result is true for all triangulations Y of S^2 with $f_0(Y) \leq n$. Let X be a triangulation of S^2 with $n+1$ vertices, which is not stacked. We consider the following cases:

Case 1: X has a vertex of degree 3. Then, from the proof of Lemma 2.9, there exists an n -vertex triangulation Y of S^2 such that X is obtained from Y by a bistellar 0-move. Since X is not stacked, it follows that Y is not stacked. Therefore, by the induction hypothesis $s(Y) < (n-8)(n+1)/20$. Then, by Lemma 3.2, we have $s(X) = (n+2)s(Y)/(n+1) + (n+2)/20 < (n-7)(n+2)/20$.

Case 2: X has a vertex of degree 4, but no vertices of degree less than 4. By the proof of Lemma 2.9, there exist vertices a, b, c, d, x and triangulations Y, Z of S^2 such that: (i) $\text{lk}_X(x) = C_4(a, b, c, d)$, (ii) $Y := X[V(X) \setminus \{x\}] \cup \{abc, adc, ac\}$ and (iii) Z is obtained from Y by starring the vertex x in the face abc , and X is obtained from Z by the 1-move $ac \mapsto dx$. By the induction hypothesis $s(Y) \leq (n-8)(n+1)/20$.

Let $U = V(X) \setminus \{a, b, c, d, x\}$. For $A \subseteq U$, let $\varphi(A \cup \{a, c\}) = A \cup \{d, x\}$. Let T^+ (resp. T^-) be those subsets S of $V(X)$ such that $q(X[S]) > q(Z[S])$ (resp., $q(X[S]) < q(Z[S])$). As removing (resp., adding) an edge increases (resp., decreases) the number of components by at most one, we have $T^+ = \{S \subseteq V(X) : q(X[S]) = q(Z[S]) + 1\}$, and $T^- = \{S \subseteq V(X) : q(X[S]) = q(Z[S]) - 1\}$. Clearly $S \in T^+ \Rightarrow \{a, c\} \subseteq S$. Similarly $S \in T^- \Rightarrow \{d, x\} \subseteq S$. We have the following cases.

Case 2a: bd is not an edge in Z . Let $A \cup \{a, c\}$ be a set in T^+ . Then A does not contain any common neighbors of a and c (otherwise $q(X[A \cup \{a, c\}]) = q(Z[A \cup \{a, c\}])$). Thus, $b, d, x \notin A$ and hence $A \subseteq U$. Then $\varphi(A \cup \{a, c\}) = A \cup \{d, x\}$. Since $Z[A \cup \{d, x\}]$ does not have a d - x path and dx is an edge in X , we have $q(X[A \cup \{d, x\}]) = q(Z[A \cup \{d, x\}]) - 1$. Therefore, φ is an injection from T^+ to T^- , which preserves sizes of sets. Then, it follows that $s(X) \leq s(Z)$. Consider the set $S = \{b, d, x\}$. Since bd is not an edge, we have $S \in T^-$. Since $S \notin \varphi(T^+)$, by Lemma 3.2, we have $s(X) < s(Z) = \frac{(n+2)s(Y)}{n+1} + \frac{n+2}{20} \leq \frac{(n+2)(n-8)}{20} + \frac{n+2}{20} = \frac{(n+2)(n-7)}{20}$.

Case 2b: bd is an edge in Z . As in Case 2a, we can show that $s(X) \leq s(Z)$. Suppose Y is a stacked sphere. Then Y has at least two vertices of degree 3. Since $d_X(u) = d_Y(u)$ for all $u \in U$, we conclude that at least two vertices among a, b, c, d have degree 3 in Y . Since $K_4 \subseteq Y[\{a, b, c, d\}]$, this implies that $Y[\{a, b, c, d\}] \cong S_4^2$. Thus, $S_4^2 \subseteq Y$. This is not possible since Y is a triangulation of S^2 and $f_0(Y) \geq 5$. Therefore, Y is not a stacked sphere and hence by the induction hypothesis for Y , $s(Y) < (n-8)(n+1)/20$. Then, by Lemma 3.2, $s(X) \leq s(Z) = \frac{(n+2)s(Y)}{n+1} + \frac{n+2}{20} < \frac{(n+2)(n-8)}{20} + \frac{n+2}{20} = \frac{(n+2)(n-7)}{20}$.

Case 3: All vertices in X have degree 5 or more. By the the proof of Lemma 2.9 there exist vertices a, b, c, d, e and x in X and triangulations Y, Z of S^2 such (i) $\text{lk}_X(x) = C_5(a, b, c, d, e)$, (ii) $Y := X[V(X) \setminus \{x\}] \cup \{abc, acd, ade, ac, ad\}$, Z is obtained from Y by starring the vertex x in the face acd , and (iii) X is obtained from Z by a $1A$ -move ($ac \mapsto bx$) followed by a $1B$ -move ($ad \mapsto ex$). Observe that for any vertex $u \in V(Y)$, $d_Y(u) \geq d_X(u) - 1$. Thus $d_Y(u) \geq 4$ for all $u \in V(Y)$. Therefore Y is not a stacked sphere. Hence by the induction hypothesis, $s(Y) < (n-8)(n+1)/20$.

We now show that $s(X) \leq s(Z)$. For $A \subseteq W := V(X) \setminus \{a, b, c, d, e, x\}$, let

$$\begin{aligned} \varphi(A \cup \{a, c, d\}) &= A \cup \{b, e, x\}, \quad \varphi(A \cup \{a, c\}) = A \cup \{b, x\}, \\ \varphi(A \cup \{a, c, e\}) &= A \cup \{c, e, x\}, \quad \varphi(A \cup \{a, d\}) = A \cup \{e, x\}, \text{ and} \\ \varphi(A \cup \{a, b, d\}) &= A \cup \{b, d, x\}. \end{aligned}$$

Let $V^+ := \{S \subseteq V(X) : q(X[S]) > q(Z[S])\}$ and $V^- := \{S \subseteq V(X) : q(X[S]) < q(Z[S])\}$. Observe that for a set $S \in V^+$, $q(X[S]) = q(Z[S]) + 1$ while for $S \in V^-$, $q(X[S]) = q(Z[S]) - 1$ or $q(Z[S]) - 2$.

First we show that every set in V^+ occurs on the left side of the mapping φ . Since the only edges of Z not present in X are ac and ad , therefore any set $S \in V^+$ contains

a and at least one of c, d . Let $S \in V^+$. Then if $\{a, c, d\} \subseteq S$, we see that $b, e, x \notin S$ (otherwise the induced subgraph on S does not decompose further on removal of edges ac and ad). Thus $S = A \cup \{a, c, d\}$ for some $A \subseteq W$. Next suppose $\{a, c\} \subseteq S$ but $d \notin S$. Then S does not contain an a - c path in X , therefore $b, x \notin S$. Thus, we have two possibilities, $S = A \cup \{a, c\}$ or $S = A \cup \{a, c, e\}$ for some $A \subseteq W$. The case $\{a, d\} \subseteq S$ but $c \notin S$ is symmetric, and leads to the last two descriptions of the sets in the mapping φ above. Hence we have shown that φ is defined on V^+ .

It is easily checked that φ is an injection that preserves sizes of sets. To establish $s(X) \leq s(Z)$ it is enough to show that $\varphi(S) \in V^-$ whenever $S \in V^+$. We argue each case separately.

Case 3a: $S = A \cup \{a, c, d\}$. Then $\varphi(S) = A \cup \{b, e, x\}$. Observe that there is no x - b or x - e path in $Z[A \cup \{b, e, x\}]$. But b, e, x lie in the same connected component in X . Thus $q(X[\varphi(S)]) < q(Z[\varphi(S)])$, and $\varphi(S) \in V^-$.

Case 3b: $S = A \cup \{a, c\}$. In this case $\varphi(S) = A \cup \{b, x\}$. Since there is no b - x path in $Z[\varphi(S)]$ and bx is an edge in X , we have $\varphi(S) \in V^-$.

Case 3c: $S = A \cup \{a, c, e\}$. Since $S \in V^+$, we conclude there is no c - e path in $X[A]$. Then $\{x, c\}$ are separated from e in Z but are in the same component of X . Therefore, $\varphi(S) = A \cup \{c, e, x\} \in V^-$.

The other two cases, namely, $S = A \cup \{a, d\}$ and $S = A \cup \{a, b, d\}$ are symmetric to the cases 3b and 3c respectively. Thus, we have $s(X) \leq s(Z)$.

Since Z is obtained from Y by a 0-move, by Lemma 3.2, we have $s(Z) = (n + 2)s(Y)/(n + 1) + (n + 2)/20$. Then, by the same argument as in Case 2b, $s(X) < (n + 2)(n - 7)/20$. This proves the theorem. \square

Corollary 3.4. *A neighborly n -vertex triangulated 3-manifold X is in the class $\mathcal{K}(3)$ if and only if $\mu_1(X) = (n - 4)(n - 5)/20$.*

Proof. Since X is an n -vertex neighborly 3-manifold each vertex-link is an $(n - 1)$ -vertex triangulation of S^2 . From Theorem 1.1, we have

$$\begin{aligned} \mu_1(X) &= 1 + \frac{1}{n} \sum_{x \in V(X)} s(\text{lk}_X(x)) \\ &\leq 1 + \frac{1}{n} \sum_{x \in V(X)} \frac{n(n - 9)}{20} = 1 + \frac{n(n - 9)}{20} = \frac{(n - 4)(n - 5)}{20}. \end{aligned} \quad (7)$$

Now, if $\mu_1(X) = (n - 4)(n - 5)/20$ then (by (7)) all the vertex-links in X satisfy Theorem 1.1 with equality, and hence are stacked spheres. So, $X \in \mathcal{K}(3)$. Conversely, if $X \in \mathcal{K}(3)$ then all vertex-links are stacked 2-spheres. Therefore, by Theorem 1.1 and (7), we get $\mu_1(X) = (n - 4)(n - 5)/20$. This proves the lemma. \square

Proof of Theorem 1.2. Let X be a tight neighborly 3-manifold with $f_0(X) = n$. Then from (1), we have $\beta_1(X) = (n - 4)(n - 5)/20$. Also, from Proposition 2.2(a), X is neighborly. Then, by (7), $(n - 4)(n - 5)/20 = \beta_1(X) \leq \mu_1(X) \leq (n - 4)(n - 5)/20$. Therefore, $\mu_1(X) = (n - 4)(n - 5)/20$. The result now follows from Corollary 3.4. \square

Proof of Corollary 1.3. Follows from Theorem 1.2 and Proposition 2.5. \square

Remark 3.5. We can give explicit examples of neighborly members of $\mathcal{K}(3)$ which are not tight neighborly, thus disproving the converse of Theorem 1.2. By Parles’s result (see [2]), a polytopal neighborly 3-sphere is in $\mathcal{K}(3)$. In particular, the boundary complex of the cyclic 4-polytope is a neighborly member of $\mathcal{K}(3)$. Therefore, by Corollary 3.4, if S is any polytopal neighborly 3-sphere with $f_0(S) \geq 6$ then S is a neighborly member of $\mathcal{K}(3)$ and $\beta_1(S) = 0 < (f_0(S) - 4)(f_0(S) - 5)/20 = \mu_1(S)$.

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